ALGORITHMIC DETECTABILITY OF IWIP AUTOMORPHISMS

ILYA KAPOVICH

Abstract. We produce an algorithm that, given $\varphi \in \mathrm{Out}(F_N)$, where $N \geq 2$, decides wether or not φ is an iwip ("fully irreducible") automorphism.

1. Introduction

The notion of a pseudo-Anosov homeomorphism of a compact surface plays a fundamental role in low-dimensional topology and the study of mapping class groups. In the context of $\mathrm{Out}(F_N)$ the concept of being pseudo-Anosov has several (non-equivalent) analogs.

The first is the notion of an "atoroidal" automorphism. An element $\varphi \in \mathrm{Out}(F_N)$ is called *atoroidal* if there do not exist $m \geq 1$, $h \in F_N, h \neq 1$ such that φ^m preserves the conjugacy class $[h]$ of h in F_N. A key result of Brinkmann [9], utilizing the Bestvina-Feighn Combination Theorem [1], says that $\varphi \in \mathrm{Out}(F_N)$ is atoroidal if and only if the mapping torus group of some (equivalently, any) representative $\Phi \in \mathrm{Aut}(F_N)$ of φ is word-hyperbolic. Another, more important, free group analog of being pseudo-Anosov is the notion of a "fully irreducible" or "iwip" automorphism. An element $\varphi \in \mathrm{Out}(F_N)$ is called *reducible* if there exists a free product decomposition $F_N = A_1 * \cdots * A_k * C$ with $k \geq 1$, $A_i \neq 1$ and $A_i \neq F_N$ such that φ permutes the conjugacy classes $[A_1], \ldots, [A_k]$. An element $\varphi \in \mathrm{Out}(F_N)$ is *irreducible* if it is not reducible. An element $\varphi \in \mathrm{Out}(F_N)$ is *fully irreducible* or *iwip* (which stands for "irreducible with irreducible powers") if φ^m is irreducible for all integers $m \geq 1$ (equivalently, for all nonzero integers m). Thus φ is an iwip if and only if there do not exist a proper free factor A of F_N and $m \geq 1$ such that $\varphi^m([A]) = [A]$. The notion of an iwip automorphism plays a key role in the study of geometry and dynamics of $\mathrm{Out}(F_N)$ and of the Culler-Vogtmann Outer space (see, for example [18, 22, 4, 14, 8, 10, 11, 16], etc).

If S is a connected compact surface, there are well-known algorithms (e.g. see [3]) to decide whether or not an element $g \in Mod(S)$ of the mapping class group of S is pseudo-Anosov. Similarly, because of the result of Brinkmann mentioned above, it is easy (at least in theory) to decide algorithmically whether an element $\varphi \in \mathrm{Out}(F_N)$ is atoroidal. Namely, we pick a representative $\Phi \in \mathrm{Aut}(F_N)$ of φ, form the mapping torus group $G = F_N \rtimes_\Phi \mathbb{Z}$ of Φ and start, in parallel, checking if G is hyperbolic (e.g. using the partial algorithm of Papasoglu [19] for detecting hyperbolicity) while at the same time looking for periodic conjugacy classes of nontrivial elements of F_N. Eventually exactly one of these procedures will terminate and we will know whether or not φ is atoroidal. A similar algorithm can be used to decide, for a closed hyperbolic surface S, if an element $g \in Mod(S)$ is pseudo-Anosov.

2000 *Mathematics Subject Classification.* Primary 20F, Secondary 57M, 37B, 37D.

The author was supported by the NSF grant DMS-0904200.

By contrast, there is no obvious approach to algorithmically deciding whether an element $\varphi \in \mathrm{Out}(F_N)$ is an iwip. In this note we provide such an algorithm:

Theorem A. *There exists an algorithm that, given $N \geq 2$ and $\varphi \in Out(F_N)$ decides whether or not φ is an iwip.*

A key step in the argument is an "if and only if" criterion of iwipness for atoroidal elements of $\mathrm{Out}(F_N)$ in terms of Whitehead graphs of train-track representatives of φ, see Proposition 4.4 below. Proposition 4.4 is similar to and inspired by Lemma 9.9 in a recent paper of Pfaff [20]; see also Proposition 5.1 in a paper of Jäger and Lustig [15] for a related criterion of iwipness. Compared to the proof of Lemma 9.9 in [20], our proof of Proposition 4.4 is more elementary and does not involve any relative train-track technology or any machinery from the Bestvina-Feign-Handel work [5] on the Tits Alternative for $\mathrm{Out}(F_N)$. However, we do utilize the notion of a "stable lamination" developed by Bestvina-Feign-Handel in [4] for iwip elements of $\mathrm{Out}(F_N)$.

To the best of our knowledge, the statement of Theorem A does not exist in the literature, although it is most likely that this result is known to some experts in the field. Since the notion of an iwip plays such a fundamental role in the study of $\mathrm{Out}(F_N)$, we think it is useful to put a proof of Theorem A in writing.

I am grateful to Martin Lustig for useful discussions and to Matt Clay for pointing out and correcting an error in the enumeration procedure in Case 2 of Theorem 4.6 in an earlier version of this paper.

2. Train-track and graph terminology

For a free group F_N (where $N \geq 2$) we fix an identification $F_N = \pi_1(R_N)$, where R_N is the N*-rose*, that is, a wedge of N circles.

We will only briefly recall the basic definitions related to train-tracks for free group automorphisms. We refer the reader to [2, 12, 7, 5, 8] for detailed background information.

2.1. **Graphs and graph-maps.** By a *graph* we mean a 1-dimensional cell-complex. For a graph Γ we refer to 0-cells of Γ as *vertices* and to open 1-cels of Γ as *topological edges*. We denote the set of vertices of Γ by $V\Gamma$ and the set of topological edges of Γ by $E_{top}\Gamma$. Each topological edge of Γ is homeomorphic to $(0, 1)$ and thus admits exactly two orientations. A topological edge with a choice of an orientation is called an *oriented edge* or just *edge* of Γ. We denote the set of oriented edges of Γ by $E\Gamma$. For an oriented edge e of Γ we denote by $o(e)$ the initial vertex of e and by $t(e)$ the terminal vertex of e; we also denote by e^{-1} the edge e with the opposite orientation. Thus $o(e^{-1}) = t(e), t(e^{-1}) = o(e)$ and $(e^{-1})^{-1} = e$.

If Γ is a graph, a *turn* in Γ is an unordered pair e, e' of oriented edges of Γ such that $o(e) = o(e')$. A turn e, e' is *degenerate* if $e = e'$ and *non-degenerate* if $e \neq e'$.

An *edge-path* in a graph Γ is a sequence $\gamma = e_1, \dots, e_n$ of $n \geq 1$ oriented edges such that $t(e_i) = o(e_{i+1})$ for all $1 \leq i < n$. We say that n is the *simplicial length* of γ and denote $|\gamma| = n$. We put $o(\gamma) := o(e_1)$, $t(\gamma) := t(e_n)$ and $\gamma^{-1} := e_n^{-1}, \dots, e_1^{-1}$. We also view a vertex v of Γ as an edge-path γ of simplicial length 0 with $o(\gamma) = t(\gamma) = v$.

If $\gamma = e_1, \dots, e_n$ is an edge-path in Γ and e, e' is a turn in Γ, we say that this turn is *contained in* γ if there exists $1 \leq i < n$ such that $e_i = e^{-1}, e_{i+1} = e'$ or $e_i = (e')^{-1}, e_{i+1} = e$.

An edge-path $\gamma = e_1, \dots, e_n$ is *tight* or *reduced* if there does not exist i such that $e_{i+1} = e_i^{-1}$, that is, if every turn contained in γ is non-degenerate.

A closed edge-path $\gamma = e_1, \dots, e_n$ is *cyclically tight* or *cyclically reduced* if every cyclic permutation of γ is tight.

If Γ_1, Γ_2 are graphs, a *graph-map* is a continuous map $f : \Gamma_1 \to \Gamma_2$ such that $f(V\Gamma_1) \subseteq V\Gamma_2$ and such that for every oriented edge e of Γ_1 its image $f(e)$ is a tight edge-path of positive simplicial length.

Every graph-map $f : \Gamma_1 \to \Gamma_2$ comes equipped with its *derivative map* $Df : E\Gamma_1 \to E\Gamma_2$: for each $e \in E\Gamma_1$ we define $(Df)(e)$ to be the initial edge of $f(e)$.

Let Γ be a finite graph and let $f : \Gamma \to \Gamma$ be a graph-map. Let $r = \#E_{top}\Gamma$ and let $E_{top}\Gamma = \{e'_1, \dots, e'_r\}$ be an ordering of the set of topological edges of Γ. For each $i = 1, \dots, r$ let e_i be an oriented edge corresponding to some choice of an orientation on the topological edge e'_i. The *transition matrix* $A(f) = (a_{ij})_{i,j=1}^r$ of f (with respect to this ordering) is an $r \times r$-matrix where the entry a_{ij} is the total number of occurrences of $e_i^{\pm 1}$ in the path $f(e_j)$. We say that $A(f)$ is *positive*, denoted $A(f) > 0$, if $a_{ij} > 0$ for all $1 \le i, j \le r$. We say that $A = A(f)$ is *irreducible* if for every $1 \le i, j \le r$ there exists $t = t(i, j) \ge 1$ such that $(A^t)_{ij} > 0$. Thus if $A(f) > 0$ then $A(f)$ is irreducible.

Recall that a vertex $v \in V\Gamma$ is *f-periodic* (or just *periodic*) if there exists $n \ge 1$ such that $f^n(v) = v$. Similarly, an edge $e \in E\Gamma$ is *f-periodic* (or just *periodic*) if there exists $n \ge 1$ such that $f^n(e)$ starts with e.

2.2. **Train-tracks.** Let Γ be a finite connected graph. A graph-map $f : \Gamma \to \Gamma$ is a *train-track map* if for every edge $e \in E\Gamma$ and for every $n \ge 1$ the path $f^n(e)$ is tight (that is, if all the turns contained in $f^n(e)$ are non-degenerate). A train-track map $f : \Gamma \to \Gamma$ is *expanding* if there exists $e \in E\Gamma$ such that $|f^n(e)| \to \infty$ as $n \to \infty$.

Remark 2.1. If $f : \Gamma \to \Gamma$ is a train-track map, then for every $m \ge 1$ the map $f^m : \Gamma \to \Gamma$ is also a train-track map. Moreover, the definition of the transition matrix implies that for every $m \ge 1$ we have $A(f^m) = [A(f)]^m$.

If $f : \Gamma \to \Gamma$ is a train-track map, we say that a turn e, e' in Γ is *taken* by f is there exist $n \ge 1$ and $e'' \in E\Gamma$ such that the turn e, e' is contained in the path $f^n(e'')$. Note that a taken turn is necessarily non-degenerate, since f is a train-track map.

Let $\varphi \in \mathrm{Out}(F_N)$. A *topological representative* of φ consists of a homotopy equivalence $\alpha : R_N \to \Gamma$ (sometimes called a *marking*), where Γ is a finite connected graph, and a graph-map $f : \Gamma \to \Gamma$ with the following properties:

(1) The map f is a homotopy equivalence.
(2) If $\beta : \Gamma \to R_N$ is a homotopy inverse of α then at the level of $F_N = \pi_1(R_N)$, the map $\beta \circ f \circ \alpha : R_N \to R_N$ induces precisely the outer automorphism φ.

For an outer automorphism $\varphi \in \mathrm{Out}(F_N)$ (where $N \ge 2$), a *train-track representative* of φ is a topological representative $f : \Gamma \to \Gamma$ of φ such that f is a train-track map, and such that every vertex of Γ has degree ≥ 3. Note that if $f : \Gamma \to \Gamma$ is a train-track representative of φ then for every $m \ge 1$ the map $f^m : \Gamma \to \Gamma$ is a train-track representative of φ^m.

An important basic result of Bestvina and Handel [2] states that every irreducible $\varphi \in \mathrm{Out}(F_N)$ (where $N \ge 2$) admits a train-track representative with an irreducible transition-matrix.

Definition 2.2 (Whitehead graph of a train-track)**.** Let $f : \Gamma \to \Gamma$ be a train-track map representing $\varphi \in \mathrm{Out}(F_N)$. Let $v \in V\Gamma$. The *Whitehead graph* $Wh_\Gamma(v, f)$ is a simple graph defined as follows. The set of vertices of $Wh_\Gamma(v, f)$ is the set of all oriented edges e of Γ with $o(e) = v$.

Two distinct oriented edges e', e'' of Γ with origin v represent adjacent vertices in $Wh_\Gamma(v, f)$ if the turn e', e'' is taken by f, that is, if there exist $e \in E\Gamma$ and $n \geq 1$ such that the turn e', e'' is contained in the edge-path $f^n(e)$.

Remark 2.3. Let $\varphi \in \mathrm{Out}(F_N)$ (where $N \geq 2$) and let $f : \Gamma \to \Gamma$ be a train-track representative such that for some $m \geq 1$ we have $A(f^m) > 0$. Then $A(f^t)$ is irreducible for all $t \geq 1$ and, moreover, $A(f^t) > 0$ for all $t \geq m$. Hence for every $v \in V\Gamma$ and $t \geq 1$ we have $Wh_\Gamma(v, f) = Wh_\Gamma(v, f^t)$.

Lemma 2.4. *Let $\varphi \in Out(F_N)$ be an iwip and let $f : \Gamma \to \Gamma$ be a train-track representative of φ. Then*

(1) *The transition matrix $A(f)$ is irreducible and for each $e \in E\Gamma$ we have $|f^n(e)| \to \infty$ as $n \to \infty$.*
(2) *There exists an integer $m \geq 1$ such that $A(f^m) > 0$.*

Proof. Part (1) is a straightforward corollary of the definitions, as observed, for example, on p. 5 of [2].

To see that (2) holds, choose $s \geq 1$ such that every periodic vertex is fixed by f^s and for every periodic edge e of Γ the path $f^s(e)$ begins with e. By part (1) we know that the length of every edge of Γ goes to infinity under the iterations of f. Hence we can find a multiple k of s such that for every edge $e \in E\Gamma$ we have $|f^k(e)| \geq 2$. Put $g = f^k$. Thus $g : \Gamma \to \Gamma$ is a train-track representative of φ^k.

Now choose a periodic edge e_0 of Γ. Since $g(e_0)$ has length ≥ 2 and starts with e_0, it follows that for every $n \geq 0$ the path $g^n(e_0)$ is a proper initial segment of $g^{n+1}(e_0)$. Let $\gamma = e_0, e_1, \dots,$ be a semi-infinite edge-path such that for all $n \geq 1$ $g^n(e_0)$ is an initial segment of γ. By construction we have $g(\gamma) = \gamma$. (That is why this γ is sometimes called a *combinatorial eigenray*, see [13]). Let $\Gamma_0 \subseteq \Gamma$ be the subgraph of Γ obtained by taking the union of all the edges of γ and their vertices. By construction $g(\Gamma_0) \subseteq \Gamma_0$ and hence $\Gamma_0 = \Gamma$ since by assumption φ is an iwip and thus φ^k is irreducible. Thus there exists $t \geq 1$ such that $g^t(e_0)$ passes through every topological edge of Γ, and therefore, for all $n \geq t$ the path $g^n(e_0)$ passes through every topological edge of Γ. Applying the same argument to every periodic edge, we can find $t \geq 1$ such that for all $n \geq t$ and every periodic edge e of Γ the path $g^n(e)$ passes through every topological edge of Γ.

Since $E\Gamma$ is finite, there is an integer $b \geq 1$ such that for every edge $e \in E\Gamma$ the initial edge of $g^b(e)$ is periodic. Then for $m = b + t$ we have $A(g^m) = A(f^{km}) > 0$, as required.

□

Remark 2.5. The proof of Lemma 2.4 can be straightforwardly modified to produce an algorithm that, given a train-track representative $f : \Gamma \to \Gamma$ of some $\varphi \in \mathrm{Out}(F_N)$ such that f satisfies condition (1) of Lemma 2.4, decides whether or not there exists $m \geq 1$ such that $A(f^m) > 0$, and if yes, produces such m. Namely, define $g = f^k$ exactly as in the proof of Lemma 2.4. Then, given a periodic edge e, start iterating g on e until the first time we find $t \geq 1$ such that $g^{t+1}(e)$ passes through the same collection of topological edges of Γ as does $g^t(e)$. Let $\Gamma_0 = \Gamma_0(e)$

be the subgraph of Γ given by the union of edges of $g^t(e)$. By construction, we have $g(\Gamma_0) \subseteq \Gamma_0$. If $\Gamma_0 \neq \Gamma$, then Γ_0 is a proper f^k-invariant subgraph of Γ and hence there does not exist $m \geq 1$ such that $A(f^m) > 0$. If for every periodic edge e we have $\Gamma_0(e) = \Gamma$, then we have found $t \geq 1$ such that for all $n \geq t$ and every periodic edge e of Γ the path $g^n(e)$ passes through every topological edge of Γ. Then, again as in the proof of Lemma 2.4, we can find an integer $b \geq 1$ such that for every edge $e \in E\Gamma$ the initial edge of $g^b(e)$ is periodic. Then for $m = b + t$ we have $A(g^m) = A(f^{km}) > 0$.

3. Stable laminations

In [4] Bestvina, Feighn and Handel defined the notion of a "stable lamination" associated to an iwip $\varphi \in \mathrm{Out}(F_N)$. A generalization of this notion for arbitrary automorphism plays a key role in the solution of the Tits Alternative for $\mathrm{Out}(F_N)$ by Bestvina, Feighn and Handel [5, 6]. We need to state their definition of a "stable lamination" in a slightly more general context than that considered in [4].

For the remainder of this section let $\varphi \in \mathrm{Out}(F_N)$ be an outer automorphism (where $N \geq 2$) and let $f : \Gamma \to \Gamma$ be a train-track representative of φ such that for some $m \geq 1$ we have $A(f^m) > 0$. (By a result of [2] and Lemma 2.4 every iwip φ admits a train-track representative with the above property, and, moreover, every train-track representative of an iwip φ has this property.)

Note that the assumption on f implies that $A(f^k)$ is irreducible for every $k \geq 1$ and, moreover, $A(f^k) > 0$ for all $k \geq m$.

Definition 3.1 (Stable lamination)**.** The *stable lamination* $\Lambda(f)$ of f consists of all the bi-infinite edge-paths

$$\gamma = \dots e_{-1}, e_0, e_1, e_2, \dots$$

in Γ with the following property:

For all $i \leq j$, $i, j \in \mathbb{Z}$ there exist $n \geq 1$ and $e \in E\Gamma$ such that $e_i, \dots, e_j$ is a subpath of the path $f^n(e)$. A path γ as above is called a *leaf* of $\Lambda(f)$.

Note that Remark 2.3 implies that, under the assumptions on f made in this section, for every $k \geq 1$ we have $\Lambda(f) = \Lambda(f^k)$.

Let $H \leq F_N$ be a nontrivial finitely generated subgroup. The Γ-*Stallings core* Δ_H corresponding to H (see [21, 17] for details) is the smallest finite connected subgraph of the covering $\widehat{\Gamma}$ of Γ corresponding to $H \leq F_N$, such that the inclusion $\Delta_H \subseteq \widehat{\Gamma}$ is a homotopy equivalence. Note that Δ_H comes equipped with a canonical immersion $\Delta_H \to \Gamma$ obtained by the restriction of the covering map $\widehat{\Gamma} \to \Gamma$ to the subgraph Δ_H. By construction every vertex of Δ_H has degree ≥ 2. Moreover, it is not hard to see that for every $w \in F_N$ we have $\Delta_H = \Delta_{wHw^{-1}}$.

We say that a nontrivial finitely generated subgroup $H \leq F_N$ *carries a leaf* of $\Lambda(f)$ if there exists a leaf γ of $\Lambda(f)$ such that γ lifts to a bi-infinite path in Δ_H.

4. Whitehead graphs and algorithmic decidability of being an iwip

The following statement, based on the procedure of "blowing up" a train-track, is fairly well-known, and first appears, in somewhat more restricted context, in the proof of Proposition 4.5 in [2]. We present a sketch of the proof for completeness.

Proposition 4.1. *Let $N \geq 2$, $\varphi \in Out(F_N)$ and let $f : \Gamma \to \Gamma$ be an expanding train-track representative of φ. Suppose that there exists a vertex $u \in V\Gamma$ such that the Whitehead graph $Wh_\Gamma(u, f)$ is disconnected. Then φ is reducible.*

Sketch of proof. We construct a graph Γ' and a graph-map $f' : \Gamma' \to \Gamma'$ as follows. For each vertex v of Γ introduce a new vertex v^* and k vertices new $v_1, \dots, v_k$ where k is the number of connected components of $Wh_\Gamma(v, f)$. We call v^* a *center-vertex* and the vertices v_i *sub-vertices*. The vertex set of Γ' consists of the center-vertices and sub-vertices corresponding to all $v \in V\Gamma$. The edge-set of Γ' is a disjoint union of two sets of edges. First, every oriented edge e of Γ is also an edge of Γ'. For $e \in E\Gamma$ with $v = o(e)$ in Γ we put $o(e) = v_i$ in Γ' where v_i is the sub-vertex coming from v corresponding to the connected component of $Wh_\Gamma(u, f)$ containing e. Second, for each $v \in V\Gamma$ with the corresponding sub-vertices $v_1, \dots, v_k$ we have an edge connecting v^* and v_i in Γ'. We call these latter types of edges of Γ' *sub-edges* corresponding to v. Note that the graph Γ' is connected but it may have degree-one vertices (namely, those center-vertices v^* such that $Wh_\Gamma(v, f)$ is connected).

We now define a map $f' : \Gamma' \to \Gamma'$. For each vertex $v \in V\Gamma$ with $z = f(v)$ put $f'(v^*) = z^*$. Let v_i be a sub-vertex corresponding to v and e is an edge of Γ originating at v and belonging to the connected component of $Wh_\Gamma(v, f)$ representing v_i. We put $f'(v_i)$ to be the sub-vertex at $z = f(v)$ corresponding to the initial edge $Df(e)$ of $f(e)$. It is easy to check that if two edges $e_1, e_2 \in E\Gamma$ with origin v are adjacent in $Wh_\Gamma(v, f)$ then the edges $Df(e_1)$ and $Df(e_2)$ are adjacent in $Wh_\Gamma(z, f)$. It follows that for any edge $e \in E\Gamma$ the edge-path $f(e)$ in Γ can also be viewed as an edge-path in Γ' and we put $f'(e) = f(e)$. Finally, if e a sub-edge at v joining v^* and a sub-vertex v_i, and if $z = f(v)$, we put $f'(e)$ to be the sub-edge joining z^* and the sub-vertex $f'(v_i)$. A straightforward check shows that $f' : \Gamma' \to \Gamma'$ is a continuous graph-map. Moreover, contracting all the sub-edges in Γ' to points is a homotopy equivalence between Γ' and Γ. Thus $f' : \Gamma' \to \Gamma'$ is a topological representative of φ.

Let Δ be the subgraph of Γ' given by the union of all the edges of Γ and of their end-vertices in Γ' (i.e. of all the sub-vertices). Thus, topologically, Δ is obtained from Γ' by removing all the center-vertices and the interiors of all the sub-edges.

By construction we have $f'(\Delta) \subseteq \Delta$. The assumption that there exists a vertex $u \in V\Gamma$ such that the Whitehead graph $Wh_\Gamma(u, f)$ is disconnected implies that the inclusion $\Delta \subseteq \Gamma'$ is not a homotopy equivalence. Moreover, the graph Δ is not a forest. Indeed, by assumption f is expanding. Choose an edge e of Γ and $n \geq 1$ such that the simplicial length of $f^n(e)$ is greater than the number of oriented edges in Γ. Then $f^n(e)$ contains an edge subpath γ such that γ is a nontrivial simple circuit in Γ. Then, by definition of Γ' and Δ, γ is also a circuit in Δ. Thus Δ is not a forest. Since Δ is f'-invariant, homotopically nontrivial, and its inclusion in Γ' is not a homotopy equivalence, we conclude that φ is reducible, as claimed.

□

Proposition 4.2. *Let $N \geq 2$, $\varphi \in Out(F_N)$ and let $f : \Gamma \to \Gamma$ be a train-track representative of φ such that $A(f) > 0$. Suppose that for every $v \in V\Gamma$ the Whitehead graph $Wh_\Gamma(v, f)$ is connected.*

Then there does not exist a finitely generated subgroup of infinite index in F_N that carries a leaf of the lamination $\Lambda(f)$.

Sketch of proof. The proof Proposition 2.4 in [4] and the proof of Lemma 2.1 in [4] on which Proposition 2.4 relies, work verbatim under the above assumptions. The conclusion of Proposition 2.4 of [4] is exactly the conclusion that we need, namely that no f.g. subgroup of infinite index in F_N carries a leaf of $\Lambda(f)$. We provide a sketch of the proof, for completeness.

Note that for any $k \ge 1$ we have $A(f^k) > 0$ and $\Lambda(f^k) = \Lambda(f)$. Thus if needed, we can always replace f by its positive power, and we will repeatedly do so below. Suppose that a leaf of $\Lambda(f)$ is carried by a finitely generated infinite index subgroup $H \le F_N$. First, by adding some edges, we complete Δ_H to a finite cover Γ_1 of Γ. Note that since H has infinite index in F_N, we really do need to add at least one new edge. We then pass to a further finite cover Γ' of Γ_1 (and thus of Γ) such that f lifts to a map $f' : \Gamma' \to \Gamma'$. Denote the covering map by $\pi : \Gamma' \to \Gamma$. By construction, f' is a train-track map and for every $k \ge 1$ $(f')^k$ is a lift of f^k. We may assume, after passing to powers, that for every f'-periodic edge e' of Γ' the path $f(e')$ begins with e', and that the same property holds for f. Obviously, every turn in Γ' taken by f' projects to a turn in Γ taken by f.

Claim 1. We claim that, after possibly replacing f' by a further power, a non-degenerate turn in Γ' is taken by f' if and only if this turn projects to a turn in Γ taken by f.

Let $a'b'$ be a reduced edge-path of length two in Γ' projecting to a path ab in Γ such that the turn a^{-1}, b is taken by f. The assumption on f implies that, after possibly passing to further powers, we have $f(a) = \dots ab \dots$. This yields a fixed point x of f in the interior of a. Since f is a homotopy equivalence, f' permutes the finite set $\pi^{-1}(x)$ in Γ'. Passing to a further power, we may assume that f' actually fixes $\pi^{-1}(x)$ pointwise. Therefore we get a fixed point of f' inside a' and, using the fact that π is a covering, we conclude that the path $f'(a')$ contains the turn $(a')^{-1}, b'$. A similar argument shows that (again after possibly taking further powers), the path $f'(b')$ also contains the turn $(a')^{-1}, b'$. This implies, in particular, that a non-degenerate turn Γ' is taken by f' if and only if this turn is a lift to Γ' of a turn taken by f, thus verifying Claim 1.

We pass to an iterate of f' for which the conclusion of Claim 1 holds, and replace f by its corresponding iterate. Then for every vertex v' of Γ' projecting to a vertex v in Γ the Whitehead graph $Wh_{\Gamma'}(v', f')$ is exactly the lift of $Wh_{\Gamma}(v, f)$, and, in particular, $Wh_{\Gamma'}(v', f')$ is connected.

Claim 2. The matrix $A(f')$ is irreducible.

Let a', b' be arbitrary edges of Γ'. Consider the maximal subgraph Γ'' of Γ' obtained as the union of all edges c' admitting an edge-path $a' = e'_0, \dots, e'_n = c'$ in Γ' such that every turn contained in this path is taken by f'. The properties of f' established in the proof of Claim 1 above imply that for every edge c' of Γ' some f'-iterate of a' passes through c'. We claim that $\Gamma'' = \Gamma'$. If not, then there exists a vertex v' of Γ which is adjacent to both Γ'' and $\Gamma' \setminus \Gamma''$. The Whitehead graph $Wh_{\Gamma'}(v', f')$ is connected, and hence there is an f'-taken turn at v' consisting of an edge of Γ'' and an edge of $\Gamma' \setminus \Gamma''$, contrary to maximality of Γ''. Thus indeed $\Gamma'' = \Gamma'$ and hence $b' \in E\Gamma'$. This means that some iterate of a' under f' passes through b'. Since a', b' were arbitrary, it follows that $A(f')$ is irreducible, and Claim 2 is established.

Recall that we assumed that the statement of the proposition fails for H, so that there exists a leaf γ of $\Lambda(f)$ that lifts to Δ_H. Choose an f-periodic edge e in γ.

Then for every $n \ge 1$ the path $f^n(e)$ lifts to a path α_n in Δ_H, and, in turn, α_n lifts to a path β_n in Γ'. Each β_n projects to $f^n(e)$ and starts with an f'-periodic edge e'_n. Since Γ' is finite, we can find a sequence $n_i \to \infty$ as $i \to \infty$ and an f'-periodic edge e' such that for all $i = 1, 2, \dots$ we have $e'_{n_i} = e'$, so that β_{n_i} starts with e'. Since e' is f'-periodic and $f'(e')$ starts with e', and since f' is a lift of f, it follows that the path $(f')^{n_i}(e') = \beta_{n_i}$ projects to a path in Δ_H for all $i \ge 1$. Since for every $s \le n_i$ $(f')^s(e')$ is an initial segment of $(f')^{n_i}(e')$, it follows that for every $n \ge 1$ the path $(f')^n(e')$ projects to an edge-path in Δ_H. Therefore for an edge e'' of Γ' covering an edge of of $\Gamma_1 \setminus \Delta_H$ there does not exist $n \ge 1$ such that $(f')^n(e')$ passes through e''. This contradicts the fact that $A(f')$ is irreducible.

□

Definition 4.3 (Clean train-track)**.** Let $f : \Gamma \to \Gamma$ be a train-track map. We say that f is *clean* if for some $m \ge 1$ we have $A(f^m) > 0$ and if for every vertex v of Γ the Whitehead graph $Wh_\Gamma(f, v)$ is connected.

Proposition 4.4. *Let $N \ge 3$ and let $\varphi \in Out(F_N)$ be an atoroidal element. Then the following conditions are equivalent:*

1. *The automorphism φ is an iwip.*
2. *There exists a clean train-track representative $f : \Gamma \to \Gamma$ of φ and, moreover, every train-track representative of φ is clean.*
3. *There exists a clean train-track representative $f : \Gamma \to \Gamma$ of φ.*

Proof. We first show that (1) implies (2). Thus suppose that φ is a atoroidal iwip. Then, as proved in [2], there exists a train-track representative of φ. Let $f : \Gamma \to \Gamma$ be an arbitrary train-track representative of φ. Since φ is an iwip, Lemma 2.4 implies that $A(f)$ is irreducible and that there exists $m \ge 1$ such that $A(f^m) > 0$. Hence, by Remark 2.3, for all $v \in V\Gamma$ and all $t \ge 1$ we have $Wh_\Gamma(f, v) = Wh_\Gamma(f^t, v)$. Moreover, Proposition 4.1 now implies that for every vertex v of Γ the Whitehead graph $Wh_\Gamma(f, v) = Wh_\Gamma(f^m, v)$ is connected. Thus f is clean and condition (2) is verified.

It is obvious that (2) implies (3). It remains to show that (3) implies (1). Thus suppose that there exists a clean train-track representative $f : \Gamma \to \Gamma$ of φ.

We claim that φ is an iwip. Suppose not. Then φ^m is not an iwip either. Thus we may assume that $m = 1$, so that $A(f) > 0$.

Then there exists a proper free factor H of F_N such that for some $k \ge 1$ we have $\varphi^k([H]) = [H]$. Let Δ_H be the Γ-Stallings core for H. Choose a nontrivial element $h \in H$ and let γ be an immersed circuit in Γ representing the conjugacy class of h. Since by assumption φ is atoroidal, the cyclically tightened length of $f^n(\gamma)$ tends to ∞ as $n \to \infty$. Let s be the simplicial length of γ, so that $\gamma = e_1 \dots e_s$. Let γ_n be the immersed circuit in Γ given by the cyclically tightened form of $f^{kn}(\gamma)$. We can obtain γ_n by cyclic tightening of the path $f^{nk}(e_1) \dots f^{nk}(e_s)$. Thus γ_n is a concatenation of $\le s$ segments, each of which is a subsegment of $f^{nk}(e)$ for some $e \in E\Gamma$. Since the simplicial length of γ_n goes to infinity as $n \to \infty$, the length of at least one of these segments tends to infinity as $n \to \infty$.

By assumption γ_n lifts to a circuit in Δ_H. Hence there exists a sequence of segments α_n in Γ such that each α_n lifts to a path in Δ_H, such that the simplicial length of α_n goes to infinity as $n \to \infty$ and such that there are $e_n \in E\Gamma$ and $t_n \ge 1$ with the property that α_n is a subpath of $f^{t_n}(e_n)$. Moreover, since $E\Gamma$ is finite, after passing to a subsequence we can even assume that $e_n = e \in E\Gamma$ for all

$n \geq 1$. By a standard compactness argument, it follows that H carries a leaf of $\Lambda(f)$, contrary to the conclusion of Proposition 4.2. Thus φ is an iwip, as claimed. $\square$

Remark 4.5. The assumption that φ be atoroidal in Proposition 4.4 is essential. One can construct $\varphi \in \mathrm{Out}(F_N)$, coming from a pseudo-Anosov homeomorphism of a surface S with ≥ 2 punctures, such that there is a clean train-track $f : \Gamma \to \Gamma$ representing φ. Then Proposition 4.2 still applies, and we do know that no leaf of $\Lambda(f)$ is carried by a finitely generated subgroup of infinite index in F_N. However, φ is not an iwip, since the peripheral curves around punctures in S represent primitive elements in F_N and thus generate cyclic subgroups that are periodic proper free factors of F_N.

A specific example of this kind is provided by Bestvina and Handel in Section 6.3 of [3] and illustrated in Figure 33 on p. 139 of [3]. In this example S is a 5-pinctured sphere, so that $\pi_1(S) = F_4$, and φ is induced by a pseudo-Anosov homeomorphism of S cyclically permuting the five punctures. The outer automorphism φ of $F_4 = F(a, b, c, d)$ is represented by $\Phi \in \mathrm{Aut}(F_4)$ given by $\Phi(a) = b$, $\Phi(b) = c$, $\Phi(c) = da^{-1}$ and $\Phi(d) = d^{-1}c^{-1}$. We can represent φ in the obvious way by a graph-map $f : \Gamma \to \Gamma$ where Γ is the wedge of four loop-edges, corresponding to a, b, c, d, wedged at a single vertex v. Then, as observed in [3] and is easy to verify directly, f is a train-track map with an irreducible transition matrix. A direct check shows that $Wh_\Gamma(v, f)$ is connected and that $A(f^6) > 0$. Thus f is a clean train-track representative of φ. However, as noted above, φ is not an iwip. Thus the element $a \in F(a, b, c, d)$ in this example corresponds to a peripheral curve on S and we see that $\Phi^5(a) = cdad^{-1}c^{-1}$, so that φ^5 preserves the conjugacy class of a proper free factor $\langle a \rangle$ of $F(a, b, c, d)$. The fact that $\Phi^5(a) = cdad^{-1}c^{-1}$ also explicitly demonstrates that φ is not atoroidal. Note also that in this example φ is irreducible but it is not an iwip, since φ^5 is reducible.

Theorem 4.6. *There exists an algorithm that, given $N \geq 2$ and $\varphi \in Out(F_N)$ decides whether or not φ is an iwip.*

Proof. We first determine whether φ is atoroidal, as follows. Let $\Phi \in \mathrm{Aut}(F_N)$ be a representative of φ and put $G = F_N \rtimes_\Phi \langle t \rangle$ be the mapping torus group of Φ. It is known, by a result of Brinkmann [9], that φ is atoroidal if and only if G is word-hyperbolic. Thus we start running in parallel the following two procedures. The first is a partial algorithm, due to Papasoglu [19], detecting hyperbolicity of G. The second procedure looks for φ-periodic conjugacy classes of elements of F_N. Eventually exactly one of these two processes will terminate and we will know whether or not φ is atoroidal.

Case 1. Suppose first that φ turns out to be atoroidal (and hence $N \geq 3$).

We then run an algorithm of Bestvina-Handel [2] which tries to construct a train-track representative of φ. As proved in [2], this algorithm always terminates and either produces a train-track representative of φ with an irreducible transition matrix or finds a reduction for φ, thus showing that φ is reducible. If the latter happens, we conclude that φ is not an iwip. Suppose now that the former happens and we have found a train-track representative $f : \Gamma \to \Gamma$ of φ with irreducible $A(f)$. We first check if it is true that for every edge e of Γ there exists $t \geq 1$ such that $|f(e)| \geq 2$. If not, we conclude, by Lemma 2.4, that φ is not an iwip. If yes, we then check, e.g. using the algorithm from Remark 2.5, if there exists

an integer $m \geq 1$ such that $A(f^m) = (A(f))^m > 0$. If no such $m \geq 1$ exists, we conclude, again by Lemma 2.4, that φ is not an iwip. Suppose now we have found $m \geq 1$ such that $A(f^m) > 0$. We then check if it is true that every vertex of Γ has a connected Whitehead graph $Wh_\Gamma(v, f)$. If not, then we conclude that φ is not an iwip, by Proposition 4.1. If yes, then f is clean and we conclude that φ is an iwip, by Proposition 4.4. Thus for an atoroidal φ we can indeed algorithmically determine whether or not φ is an iwip.

Case 2. Suppose now that φ turned out to be non-atoroidal. Then Proposition 4.5 of [2] implies that φ is an iwip if and only if φ is induced by a pseudo-Anosov homeomorphism of a compact surface S with a single boundary component. Thus either φ has a periodic conjugacy class of a proper free factor of F_N or φ is induced by a pseudo-Anosov of a compact surface S with a single boundary component.

We now start running in parallel the following two processes.

The first process looks for a periodic conjugacy class of a proper free factor of F_N: we start enumerating all the proper free factors $H_1, H_2, \dots$ of F_N and for each H_i we start listing its images $\varphi(H_i), \varphi^2(H_i), \varphi^3(H_i), \dots$ and check if $\varphi^j([H_i]) = [H_i]$. The process terminates if we find i, j such that $\varphi^j([H_i]) = [H_i]$.

The second process looks for the relization of φ as a a pseudo-Anosov homeomorphism of a compact surface S as above. Note that if g belongs to the mapping class group $Mod(S)$ of S and if $\alpha_1, \alpha_2 : F_N \to \pi_1(S)$ are two isomorphisms, then the elements of $\mathrm{Out}(F_N)$ corresponding to g via α_1 and α_2 are related by a conjugation in $\mathrm{Out}(F_N)$. Thus, in order to account for all possible realizations of φ of the above type, do the following. Depending on the rank N of F_N, there are either exactly one (non-orientable) or exactly two (one orientable and one non-orientable) topological types of compact connected surfaces S with one boundary component and with $\pi_1(S)$ free of rank N. For each of these choices of S we fix an isomorphism $\alpha : F_N \to \pi_1(S)$. Then start enumerating all the elements $g_1, g_2, \dots$ of $Mod(S)$, and, for each such g_i, start enumerating all the $\mathrm{Out}(F_N)$-conjugates ψ_{ij}, $j = 1, 2, \dots$ of the element of $\mathrm{Out}(F_N)$ corresponding to g_i via α. Then for each ψ_{ij} check if $\psi_{ij} = \varphi$ in $\mathrm{Out}(F_N)$. If not, continue the enumeration of all the the ψ_{ij}, and if yes, use the algorithm from [3] to decide whether or not g_i is pseudo-Anosov. If g_i is pseudo-Anosov, we terminate the process; otherwise, we continue the diagonal enumeration of all the ψ_{ij}.

Eventually exactly one of these two processes will terminate. If the first process terminates, we conclude tha φ is not an iwip. If the second process terminates, we conclude that φ is an iwip.

$\square$

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu